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***Abstract*—This brief note proves in a direct way that two different side conditions, which have been used in the literature to characterize strictly positive real matrix transfer functions in the frequency domain, are equivalent.**

I. Introduction

The frequency domain conditions characterizing the fact that a matrix transfer function F is strictly positive real involve a positivity constraint at infinite frequency. This constraint — usually referred to as *side condition* — has been a source of confusion and controversy in the literature for more than a decade. As pointed out in [3], the side conditions used in [6], [7], [8] were incorrect as they had some inconsistencies. To fix the problem, [3] proposed a new condition at infinite frequency, i.e.

$$\lim_{\omega\to\infty} \omega^{2\rho} \det\left(F(j\omega)+F(-j\omega)^\top\right) > 0 \tag{1}$$

where ρ is the dimension of $\ker\left(F(\infty)+F(\infty)^\top\right)$.

On the other hand, a different, but equally valid, condition at infinite frequency was proposed the second edition of the book by Khalil published in 1996 (see [5, Lemma 10.1]); such a condition, that reads as follows, was recently used in [4] to establish a counterpart result for negative imaginary systems (see [4, Remark 1]): $\exists\, \delta>0, \sigma_0>0$ such that

$$\underline{\sigma}\left[\omega^2\left(F(j\omega)+F(-j\omega)^\top\right)\right] \geq \sigma_0 \quad \forall|\omega|\geq\delta. \tag{2}$$

This note is devoted to the analysis of the two side-conditions (1) and (2). We will prove that while they are in general not equivalent at infinite frequency, they are indeed equivalent under the other conditions guaranteeing that F is strictly positive real. Hence both conditions at infinite frequency are equally valid. While this could be deduced from [3] and [5], our results provide a direct proof of such equivalency.

Notation: Let the set of real (resp. complex) numbers be denoted by $\mathbb{R}$ (resp. $\mathbb{C}$) and the corresponding sets of matrices of dimension $m\times n$ be denoted by $\mathbb{R}^{m\times n}$ (resp. $\mathbb{C}^{m\times n}$). Given $M\in\mathbb{C}^{m\times m}$, $M^\star$ denotes the complex conjugate transpose of matrix M (*i.e.*, if $M=A+jB$ for real matrices A and B, then $M^\star=A^\top-jB^\top$). A matrix M is said to be Hermitian if $M=M^\star$ and $M>0$ denotes that the matrix M is Hermitian and positive definite. The smallest singular value of M is denoted by $\underline{\sigma}(M)$. We recall that the singular values of a positive semi-definite Hermitian matrix are its nonzero eigenvalues [2, p.649].

II. Main result

The following definition, adapted from [4, Definitions 1 and 2], is the standard definition for strictly positive real systems. It essentially states that a transfer function matrix $F(s)$ is strictly positive real if for some $\epsilon>0$, the transfer function matix $F(s-\epsilon)$ is positive real and $F(s)+F(-s)^\top$ has full normal rank. See also [4, Lemma 2] for an equivalent re-characterization.

Definition 2.1: Let $F:\mathbb{C}\longrightarrow\mathbb{C}^{m\times m}$ be a real transfer function. Then $F(s)$ is said to be Strictly Positive Real (SPR) if there exists a real scalar $\epsilon>0$ such that $F(s)$ is analytic in $\{s\in\mathbb{C}:\mathfrak{Re}\{s\}>-\epsilon\}$, $F(s)+F(s)^\star\geq 0$ for all $s\in\{s\in\mathbb{C}:\mathfrak{Re}\{s\}>-\epsilon\}$ and $F(s)+F(-s)^\top$ has full normal rank.

SPR matrix transfer functions can be characterized in the frequency domain by three conditions: the first two are conditions 1 and 2 in the next proposition, the third is the *side condition* and it has been stated in two different manners: side condition (3a) in Proposition 2.1 can be found in [5], [4], while side condition (3b) can be found in [3]. These side conditions can be interpreted as apparently different conditions on how $\left(F(j\omega)+F(-j\omega)^\top\right)$ approaches zero for sufficiently large $|\omega|$ in directions where it looses rank.

Proposition 2.1: Let $F:\mathbb{C}\longrightarrow\mathbb{C}^{m\times m}$ be a real, rational, proper transfer function such that the following two conditions hold:

1) $F(s)$ has no poles in $\{s\in\mathbb{C}:\mathfrak{Re}\{s\}\geq 0\}$;
2) $F(j\omega)+F(-j\omega)^\top>0$ for all $\omega\in\mathbb{R}$.

Then, the following two side conditions are equivalent:

3a) $\exists\delta>0, \sigma_0>0:$

$$\underline{\sigma}\left[\omega^2\left(F(j\omega)+F(-j\omega)^\top\right)\right]\geq\sigma_0 \quad \forall|\omega|\geq\delta \tag{3}$$

3b) $$\lim_{\omega\to\infty}\omega^{2\rho}\det[F(j\omega)+F(-j\omega)^\top]\neq 0 \tag{4}$$

where $\rho=\dim\ker(F(\infty)+F(\infty)^\top)$.

Proof. Since $F(s)$ is proper, let $F(s)=C(sI-A)^{-1}B+D$ for some state-space realization (A,B,C,D). Near $s=\infty$, the expansion

$$F(s)=F_0+F_1/s+F_2/s^2+\dots \tag{5}$$

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holds (with $F_0 = F(\infty) = D$), so that

$$F(j\omega) + F(-j\omega)^\top = Q + \frac{H}{\omega} + \frac{K(\omega)}{\omega^2} \tag{6}$$

with $Q = D + D^\top$, $H = (F_1 - F_1^\top)/j$ and $K(\omega) = \sum_{i=1}^{\infty} \left(\frac{F_{i+1}}{j^{i+1}\omega^{i-1}} + \frac{F_{i+1}^T}{(-j)^{i+1}\omega^{i-1}} \right)$. Then it is easy to see that

$$\lim_{\omega\to\infty} K(\omega) = K_0 \in \mathbb{R}^{m\times m},$$

with $K_0 := -F_2 - F_2^T$. If Q is positive definite, in view of the Hermitian symmetry of both H and $K(\omega)$, both conditions 3a) and 3b) are clearly satisfied and the result is obvious.

Assume than that Q is singular with rank $m-\rho$. Since Q is symmetric, there exists an orthogonal matrix U such that $UQU^\top = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $Q_1 \in \mathbb{R}^{(m-\rho)\times(m-\rho)}$ nonsingular. Hence, without changing the essence of the problem, we assume that Q has the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

with $Q_1 \in \mathbb{R}^{(m-\rho)\times(m-\rho)}$ nonsingular, and hence $Q_1 > 0$. We partition H and $K(\omega)$ accordingly, as follows:

$$H = \begin{bmatrix} H_1 & H_{12} \\ H_{21} & H_2 \end{bmatrix}, \qquad K(\omega) = \begin{bmatrix} K_1(\omega) & K_{12}(\omega) \\ K_{21}(\omega) & K_2(\omega) \end{bmatrix}. \tag{8}$$

Thus

$$[F(j\omega) + F(-j\omega)^\top]_{22} = \frac{1}{\omega}(H_2 + \frac{K_2(\omega)}{\omega}). \tag{9}$$

Since $H = (F_1 - F_1^\top)/j$, we have that: (i) H is Hermitian, so that H_2 is Hermitian as well, and (ii) the elements on the diagonal of H are zero, so that also the elements on the diagonal of H_2 are zero and hence H_2 is traceless. Therefore, H_2 has only real eigenvalues and the sum of the eigenvalues of H_2 is zero. Hence, either $H_2 = 0$ or H_2 has at least a negative eigenvalue. But for ω sufficiently large, and by continuity of the eigenvalues as functions of ω, the eigenvalues of $H_2 + \frac{K_2(\omega)}{\omega}$ are arbitrarily close to those of H_2, so that if $H_2 \neq 0$, then $[F(j\omega)+F(-j\omega)^\top]_{22}$ has a negative eigenvalue for a sufficiently large ω, and this is against our assumptions because $[F(j\omega) + F(-j\omega)^\top] > 0$ for all $\omega \in \mathbb{R}$ so, in turn, $[F(j\omega) + F(-j\omega)^\top]_{22} > 0$ for all $\omega \in \mathbb{R}$. In conclusion, $H_2 = 0$.
Now let

$$\Phi(\omega) = \begin{bmatrix} \Phi_1(\omega) & \Phi_{12}(\omega) \\ \Phi_{21}(\omega) & \Phi_2(\omega) \end{bmatrix} = F(j\omega) + F(-j\omega)^\top.$$

By continuity, as $\omega \to \infty$, $m-\rho$ of the eigenvalues of Φ, *i.e.*, the eigenvalues of $\Phi_1(j\omega)$, tend to the eigenvalues of Q_1 (that are strictly positive) and the remaining ρ eigenvalues tend to zero. Let $\lambda(\omega)$ be one of the eigenvalues of Φ that tends to zero as $\omega \to \infty$. We now show that $\lambda(\omega)$ tends to zero at least as fast as $1/\omega^2$. In fact, provided that ω is large enough so that $\lambda(\omega)$ is not an eigenvalue of $\Phi_1(\omega)$, then $\lambda(\omega)$ must be an eigenvalue of

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[\Phi_1(\omega) - \lambda(\omega)I]^{-1}\Phi_{12}(\omega) \tag{10}$$

because through Schur complements (e.g. [1])

$$\det[\Phi(\omega) - \lambda(\omega)I] = \det[\Phi_1(\omega) - \lambda(\omega)I]\det[R(\omega) - \lambda(\omega)I].$$

Using the fact that $\Phi_1(\omega) - \lambda(\omega)I = Q_1 + \frac{H_1}{\omega} + \frac{K_1(\omega)}{\omega^2} - \lambda(\omega)I$, and using [1, Fact 9.9.43], $R(\omega)$ in (10) can be written equivalently as

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[Q_1^{-1} + \Delta(\omega)]\Phi_{12}(\omega),$$

where $\Delta(\omega) \to 0$ as $\omega \to \infty$, since $\lambda(\omega) \to 0$ as $\omega \to \infty$. Finally, using (6) (7) and (8), we have $R(\omega) = \frac{1}{\omega^2}[K_2(\omega) - P(\omega)]$ where

$$P(\omega) := (H_{21} + \frac{K_{21}(\omega)}{\omega})[Q_1^{-1} + \Delta(\omega)](H_{12} + \frac{K_{12}(\omega)}{\omega}).$$

Since for ω sufficiently large $[K_2(\omega) - P(\omega)]$ is bounded, then the eigenvalues of $R(\omega)$ tend to zero at least as fast as $1/\omega^2$.

Then $\Phi(\omega)$ has m strictly positive eigenvalues for all $\omega \in \mathbb{R}$, ρ of which are going down to zero at least as fast as $1/\omega^2$ as $\omega \to \infty$. The remaining $(m-\rho)$ eigenvalues tend to the strictly positive eigenvalues of Q_1 as $\omega \to \infty$. Now, order the eigenvalues of $\Phi(\omega)$ in non-decreasing size.

Since $\det(\cdot)$ is the product of eigenvalues, then $\omega^{2\rho}\det[\Phi(\omega)]$ is the same as the product of $(\omega^2\lambda_1), \ldots, (\omega^2\lambda_\rho), \lambda_{\rho+1}, \ldots, \lambda_m$ because for each $i \in \{1, \ldots, \rho\}$, λ_i tends to zero at least as fast as $1/\omega^2$. Then, we have that the side condition (3b):

$$\lim_{\omega\to\infty} w^{2\rho}\det[\Phi(\omega)] \neq 0$$

is equivalent to (since $\lambda_{\rho+1}$,...,λ_m do not tend to zero)

$$\lim_{\omega\to\infty}(w^2\lambda_i) \neq 0 \text{ for all } i \in \{1, ..., \rho\},$$

which is equivalent to

$$\lim_{\omega\to\infty} w^2\lambda_1 > 0,$$

which, finally, is equivalent to the first side condition (3a). The proposition is proved. □

Examples. Consider the case of $F(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix}$. In this case $\rho = 1$ and, by direct computation, we see that

$$\lim_{\omega\to\infty} \omega^{2\rho}\det[F(j\omega)+F(-j\omega)^\top] = \lim_{\omega\to\infty} \frac{8\omega^2 + 4\omega^4}{(1+\omega^2)^2} = 4 \neq 0.$$

On the other hand, we have

$$\underline{\sigma}\left[\omega^2\left(F(j\omega) + F(-j\omega)^\top\right)\right] = \frac{2\omega^2}{1+\omega^2}$$

which is clearly greater than 1 for all $|\omega| > 1$.

The pathological case, corresponding to the situation in which some of the eigenvalues of the spectrum go to zero faster than $\frac{1}{\omega^2}$, as ω tends to infinity, is more interesting: let $F(s) = \begin{bmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+2}{(s+1)} \end{bmatrix}$. In this case $\rho = 1$ and by direct computation, we easily see that

$$\lim_{\omega\to\infty} \omega^{2\rho}\det[F(j\omega)+F(-j\omega)^\top] = \lim_{\omega\to\infty} \frac{\omega^2(16+8\omega^2)}{(1+\omega^2)^3} = 0.$$

The same conclusion is obtained with the other side condition which, in this case, reads:

$$\underline{\sigma}\left[\omega^2\left(F(j\omega)+F(-j\omega)^\top\right)\right]=\frac{4\omega^2}{(1+\omega^2)^2}$$

which is clearly not bounded away from zero as $|\omega|$ diverges.

Finally, the trivial example $F(s)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ immediately shows that, if we do not assume the first two conditions of Proposition 2.1, the two side conditions are not necessarily equivalent.

III. Conclusion

This note shows that two different side conditions used in the control literature to characterize strictly positive real matrix transfer functions are equivalent.

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